

AMALGAMS, CONNECTIFICATIONS, AND HOMOGENEOUS COMPACTA

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ABSTRACT. We construct a path-connected homogenous compactum with cellularity \mathfrak{c} that is not homeomorphic to any product of dyadic compacta and first countable compacta. We also prove some closure properties for classes of spaces defined by various connectifiability conditions. One application is that every infinite product of infinite topological sums of T_i spaces has a T_i pathwise connectification, where $i \in \{1, 2, 3, 3\frac{1}{2}\}$.

1. INTRODUCTION

In [14], M. A. Maurice constructed a family of homogeneous compact ordered spaces with cellularity \mathfrak{c} . All these spaces are zero-dimensional. The cone over any of these spaces is path-connected but not homogeneous or ordered. Indeed, it is easy to see that no compact ordered space with uncountable cellularity can be path-connected. However, there is a path-connected homogeneous compactum with cellularity \mathfrak{c} which, though not an ordered space, has small inductive dimension 1; we construct such a space by amalgamating copies of powers of one of Maurice's spaces together. Moreover, this space is not homeomorphic to a product of dyadic compacta and first countable compacta. To the best of the author's knowledge, there is only one other example [15] of a homogeneous compactum not homeomorphic to such a product, and that example's existence is independent of ZFC.

This amalgamation technique also can be used to construct new connectifications, where a connected (path-connected) space Y is a *connectification* (*pathwise connectification*) of a space X if X can be densely embedded in Y , and the connectification is *proper* if the embedding can be chosen not to be surjective. Whether a space has a connectification is uninteresting unless we restrict to connectifications that are at least T_2 . For a broad survey of connectification results, see [17]. Our focus will be on which a T_2 (T_3 , $T_{3\frac{1}{2}}$, metric) spaces have T_2 (T_3 , $T_{3\frac{1}{2}}$, metric) connectifications or pathwise connectifications. Only partial characterizations are known. For example, Watson and Wilson [16] showed that a countable T_2 space has a T_2 connectification iff it has no isolated points. Emeryk and Kulpa [6] proved that the Sorgenfrey line has a T_2 connectification, but no T_3 connectification. Alas *et al* [1] showed that every separable metric space without nonempty open compact subsets has a metric connectification. Gruenhage, Kulesza, and Le Donne [11] showed that every nowhere locally compact metric space has a metric connectification.

There are only a handful of results about pathwise connectifications. For example, Fedeli and Le Donne [9] showed that a nonsingleton countable first countable

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T_2 space has a T_2 pathwise connectification iff it has no isolated points. Druzhinina and Wilson [4] showed that a metric space has a metric pathwise connectification if its path components are open and not locally compact; similarly, a first countable T_2 (T_3) space has a T_2 (T_3) connectification if its path components are open and not locally feebly compact. See also [2] for some results about pathwise connectifications of spaces adjoined with a free open filter.

Suppose $i \in \{1, 2, 3, 3\frac{1}{2}\}$ and X has a proper T_i connectification. Then $X \times Z$ has a proper T_i connectification for all T_i spaces Z . Thus, given one proper connectification, this product closure property gives us a new connectification. We omit the easy proof of this fact here because we shall prove much stronger amalgam closure properties, which in many cases are also valid for pathwise connectifications. The reals are a pathwise connectification of the Baire space ω^ω because $\omega^\omega \cong \mathbb{R} \setminus \mathbb{Q}$. By applying amalgam closure properties to this particular connectification, we shall prove the following theorem.

Theorem 1.1. *If $i \in \{1, 2, 3, 3\frac{1}{2}\}$, then every infinite product of infinite topological sums of T_i spaces has a T_i pathwise connectification. Every countably infinite product of infinite topological sums of metrizable spaces has a metrizable pathwise connectification.*

The previously known result most similar to Theorem 1.1 is due to Fedeli and Le Donne [8]: a product of T_2 spaces with open components has a T_2 connectification iff it does not contain a nonempty proper open subset that is H -closed.

2. AMALGAMS

For all undefined notions, see [7, 12].

Definition 2.1. Given a topological space X , let $\mathbb{S}(X)$ denote the set of all subbases of X that do not include \emptyset .

Let X be a nonempty T_0 space and let $\mathcal{S} \in \mathbb{S}(X)$. For each $S \in \mathcal{S}$, let Y_S be a nonempty topological space. The *amalgam* of $\langle Y_S : S \in \mathcal{S} \rangle$ is the set Y defined by

$$Y = \bigcup_{p \in X} \prod_{p \in S \in \mathcal{S}} Y_S.$$

We say that X is the *base space* of Y . For each $S \in \mathcal{S}$, we say that Y_S is a *factor* of Y . Every amalgam has a natural projection π to its base space: because X is T_0 , we may define $\pi: Y \rightarrow X$ by $\pi^{-1}\{p\} = \prod_{p \in S \in \mathcal{S}} Y_S$ for all $p \in X$. Amalgams also have natural partial projections to their factors: for each $S \in \mathcal{S}$, define $\pi_S: \pi^{-1}S \rightarrow Y_S$ by $y \mapsto y(S)$.

Consider sets of the form $\pi_S^{-1}U$ where $S \in \mathcal{S}$ and U open in Y_S . We say such sets are *subbasic* and finite intersections of such sets are *basic*. We topologize Y by declaring these basic sets to be a base of open sets. Let us list some easy consequences of this topologization.

- (1) For all $S \in \mathcal{S}$, the map π_S is continuous and open and has open domain.
- (2) The map π is continuous and open.
- (3) If $|Y_S| = 1$ for all $S \in \mathcal{S}$, then $Y \cong X$.
- (4) For each $p \in X$, the product topology of $\prod_{p \in S \in \mathcal{S}} Y_S$ is also the subspace topology inherited from Y .

- (5) Suppose, for each $S \in \mathcal{S}$, that Z_S is a subspace of Y_S . Then the topology of the amalgam of $\langle Z_S : S \in \mathcal{S} \rangle$ is also the subspace topology inherited from Y .
- (6) Suppose, for each $S \in \mathcal{S}$, that \mathcal{S}_S is a subbase of Y_S . Then the set

$$\{\pi_S^{-1}T : S \in \mathcal{S} \text{ and } T \in \mathcal{S}_S\}$$

is a subbase of Y .

Up to homeomorphism, an amalgam is a quotient of the product of its base space and its factors. Specifically, the map from $X \times \prod_{S \in \mathcal{S}} Y_S$ to Y given by

$$\langle x, y \rangle \mapsto y \upharpoonright \{S \in \mathcal{S} : x \in S\}$$

is easily verified to be a quotient map.

We say that a class \mathcal{A} of nonempty T_0 spaces is *amalgamative* if an amalgam is always in \mathcal{A} if its base space and all its factors are in \mathcal{A} . Therefore, any class of nonempty T_0 spaces closed with respect to products and quotients is amalgamative. In particular, amalgams preserve compactness, connectedness, and path-connectedness. The next theorem says that several other well-known productive classes are also amalgamative.

Theorem 2.2. *The classes listed below are amalgamative provided we exclude the empty space. Conversely, if an amalgam is in one of these classes, then its base space and all its factors are also in that class.*

- (1) T_0 spaces
- (2) T_1 spaces
- (3) T_2 spaces
- (4) T_3 spaces
- (5) $T_{3\frac{1}{2}}$ spaces
- (6) hereditarily disconnected T_0 spaces
- (7) zero-dimensional T_0 spaces

Proof. For (1)-(3), suppose y_0 and y_1 are distinct elements of Y . If $\pi(y_0) = \pi(y_1)$, then there exists $S \in \text{dom } y_0 = \text{dom } y_1$ such that $y_0(S) \neq y_1(S)$; whence, if U_0 and U_1 are neighborhoods of $y_0(S)$ and $y_1(S)$ witnessing the relevant separation axiom for $y_0(S)$ and $y_1(S)$, then $\pi_S^{-1}U_0$ and $\pi_S^{-1}U_1$ witness the the same separation axiom for y_0 and y_1 . If $\pi(y_0) \neq \pi(y_1)$, then let U_0 and U_1 be neighborhoods of $\pi(y_0)$ and $\pi(y_1)$ witnessing the relevant separation axiom for $\pi(y_0)$ and $\pi(y_1)$. Then $\pi^{-1}U_0$ and $\pi^{-1}U_1$ witness the same separation axiom for y_0 and y_1 .

For (4) and (5), suppose C is a closed subset of Y and $y \in Y \setminus C$. Then there exist $n < \omega$ and $\langle S_i \rangle_{i < n} \in (\text{dom } y)^n$ and $\langle U_i \rangle_{i < n}$ such that U_i is a neighborhood of $y(S_i)$ for all $i < n$ and $\bigcap_{i < n} \pi_{S_i}^{-1}U_i$ is disjoint from C . For each $i < n$, let V_i be a neighborhood of $y(S_i)$ such that $\overline{V_i} \subseteq U_i$. Let U be a neighborhood of $\pi(y)$ such that $\overline{U} \subseteq \bigcap_{i < n} S_i$. Set $V = \pi^{-1}U \cap \bigcap_{i < n} \pi_{S_i}^{-1}V_i$. Then V is a neighborhood of y and we have

$$\overline{V} \subseteq \bigcap_{i < n} \pi^{-1}S_i \cap \bigcap_{i < n} \pi_{S_i}^{-1}U_i = \bigcap_{i < n} \pi_{S_i}^{-1}U_i;$$

whence, \overline{V} is disjoint from C .

Now suppose there is a continuous map $f: X \rightarrow [0, 1]$ such that $f(\pi(y)) = 1$ and $f''(X \setminus U) = \{0\}$. For each $i < n$, likewise suppose there is a continuous map $f_i: Y_{S_i} \rightarrow [0, 1]$ such that $f_i(y(S_i)) = 1$ and $f''(Y_{S_i} \setminus U_i) = \{0\}$. Define

$g: \bigcap_{i < n} \pi^{-1}S_i \rightarrow [0, 1]$ by $z \mapsto f(\pi(z))f_0(z(S_0)) \cdots f_{n-1}(z(S_{n-1}))$. Define $h: \pi^{-1}(X \setminus \overline{U}) \rightarrow [0, 1]$ by $z \mapsto 0$. By the pasting lemma, $g \cup h$ is continuous and separates y and C .

For (6), suppose C is a nonempty connected subset of Y and X and Y_S are hereditarily disconnected for all $S \in \mathcal{S}$. Then π^*C is connected; whence, $\pi^*C = \{p\}$ for some $p \in X$. For each $S \in \mathcal{S}$, if $p \in S$, then π_S^*C is connected; whence, $|\pi_S^*C| = 1$. Thus, $|C| = 1$.

For (7), suppose $S \in \mathcal{S}$ and U open in Y_S and $y \in \pi_S^{-1}U$. Let V be a clopen neighborhood of $y(S)$ contained in U . Then $\pi_S^{-1}V$ is clopen in $\pi^{-1}S$. Let W be a clopen neighborhood of $\pi(y)$ contained in S . Then $\pi^{-1}W \cap \pi_S^{-1}V$ is a clopen neighborhood of y contained in $\pi_S^{-1}U$.

For the converse, first note that each of the classes (1)-(7) is closed with respect to subspaces. Second, Y_S can be embedded in Y for all $S \in \mathcal{S}$ because $\prod_{p \in S \in \mathcal{S}} Y_S$ is a subspace of Y for all $p \in X$. Finally, X can be embedded in Y because the amalgam of $\{f(S)\}_{S \in \mathcal{S}}$ is homeomorphic to X for all $f \in \prod_{S \in \mathcal{S}} Y_S$. \square

A countable product of metrizable spaces is metrizable; the next theorem is the analog for amalgams.

Theorem 2.3. *Suppose X and Y_S are metrizable for all $S \in \mathcal{S}$ and there is a countable $\mathcal{T} \subseteq \mathcal{S}$ such that $|Y_S| = 1$ for all $S \in \mathcal{S} \setminus \mathcal{T}$. Then Y is metrizable.*

Proof. Since Y is T_3 by Theorem 2.2, it suffices to exhibit a σ -locally finite base for Y . For each $T \in \mathcal{T}$, let $\bigcup_{n < \omega} \mathcal{U}_{T,n}$ be a σ -locally finite base for Y_T ; let $\bigcup_{n < \omega} \mathcal{U}_n$ be a σ -locally finite base for X . For each $n < \omega$ and $\tau \in \text{Fn}(\mathcal{T}, \omega)$, set $\mathcal{U}_{n,\tau} = \{U \in \mathcal{U}_n : \overline{U} \subseteq \bigcap \text{dom } \tau\}$ and

$$\mathcal{V}_{n,\tau} = \left\{ \pi^{-1}U \cap \bigcap_{T \in \text{dom } \tau} \pi_T^{-1}U_T : U \in \mathcal{U}_{n,\tau} \text{ and } (\forall T \in \text{dom } \tau)(U_T \in \mathcal{U}_{T,\tau(T)}) \right\}.$$

Then $\bigcup_{n < \omega} \bigcup_{\tau \in \text{Fn}(\mathcal{T}, \omega)} \mathcal{V}_{n,\tau}$ is easily verified to be a σ -locally finite base for Y . \square

In general, productiveness is logically incomparable to amalgamateness: the class of finite T_0 spaces is amalgamative but only finitely productive; the class of powers of 2 is productive but not amalgamative. However, all amalgamative classes are finitely productive because if $X \in \mathcal{S}$ and $|Y_S| = 1$ for all $S \in \mathcal{S} \setminus \{X\}$, then $Y \cong X \times Y_X$.

Given Theorem 2.2, it is tempting to conjecture that amalgams are really subspaces of products in disguise. This conjecture is false. To see this, consider the class of nonempty Urysohn spaces. This class is closed with respect to arbitrary products and subspaces, yet, as demonstrated by the following example, this class is not amalgamative.

Example 2.4. Let $X = \mathbb{Q}$ with the topology generated by $\{\mathbb{Q} \setminus K\}$ and the order topology of \mathbb{Q} where $K = \{2^{-n} : n < \omega\}$. Then X is Urysohn. Let $\mathbb{Q} \setminus K \in \mathcal{S}$ and, for all $S \in \mathcal{S}$, let $|Y_S| = 1$ if $S \neq \mathbb{Q} \setminus K$. Set $Y_{\mathbb{Q} \setminus K} = 2$ (with the discrete topology). Then all the factors of Y are Urysohn. For each $i < 2$, define $y_i \in Y$ by $\{y_i\} = \pi^{-1}\{0\} \cap \pi_{\mathbb{Q} \setminus K}^{-1}\{i\}$. Suppose U_0 and U_1 are disjoint closed neighborhoods of y_0 and y_1 , respectively. Then π^*U_0 and π^*U_1 are neighborhoods of 0. Therefore, $2^{-n} \in \pi^*U_0 \cap \pi^*U_1$ for some $n < \omega$. If $2^{-n} \in S \in \mathcal{S}$, then $|Y_S| = 1$; hence, $\{\pi^{-1}S : 2^{-n} \in S \in \mathcal{S}\}$ is a local subbase for y_2 where $\{y_2\} = \pi^{-1}\{2^{-n}\}$. Since $2^{-n} \in \pi^*U_0 \cap \pi^*U_1$, every finite intersection of elements of this local subbase will

intersect U_0 and U_1 . Hence, $y_2 \in \overline{U_0} \cap \overline{U_1} = U_0 \cap U_1$, which is absurd. Therefore, Y is not Urysohn.

In the above example, the base space and all the factors of Y are totally disconnected. Therefore, no amalgamative class both contains all the nonempty totally disconnected spaces and is contained in the class of nonempty Urysohn spaces.

Question 2.5. Is the class of nonempty realcompact spaces amalgamative?

Despite Example 2.4, there is a sense in which Y is almost homeomorphic to a subspace of the product of its factors. For each $S \in \mathcal{S}$, let Z_S be Y_S with an added point q_S whose only neighborhood is Z_S . Then Y is easily seen to be homeomorphic to the set

$$\bigcup_{p \in X} \left\{ z \in \prod_{S \in \mathcal{S}} Z_S : (\forall S \in \mathcal{S})(z(S) = q_S \Leftrightarrow p \notin S) \right\}$$

with the subspace topology inherited from $\prod_{S \in \mathcal{S}} Z_S$. Moreover, this result still holds if we make q_S isolated for all clopen $S \in \mathcal{S}$.

Let us make some auxillary definitions relating amalgams to continuous maps and subspaces.

Definition 2.6. Suppose, for each $S \in \mathcal{S}$, that Z_S is a nonempty space and $f_S: Y_S \rightarrow Z_S$. Let Z be the amalgam of $\langle Z_S \rangle_{S \in \mathcal{S}}$. Then the *amalgam* of $\langle f_S \rangle_{S \in \mathcal{S}}$ is the map f defined by

$$f = \bigcup_{p \in X} \prod_{p \in S \in \mathcal{S}} f_S.$$

In the above definition, it is immediate that f is a map from Y to Z . Moreover, if f_S is continuous for each $S \in \mathcal{S}$, then f is a continuous map from Y to Z . Similarly, an amalgam of homeomorphisms is a homeomorphism.

Definition 2.7. Suppose W is a subspace of X . The *reduced amalgam* of $\langle Y_S \rangle_{S \in \mathcal{S}}$ over W is the space Z defined as follows. Set $\mathcal{T} = \{S \cap W : S \in \mathcal{S}\} \setminus \{\emptyset\}$. Then $\mathcal{T} \in \mathbb{S}(W)$. Given $S_0, S_1 \in \mathcal{S}$, declare $S_0 \sim S_1$ if $S_0 \cap W = S_1 \cap W$. For each $T \in \mathcal{T}$, let $\varepsilon(T)$ be the unique \mathcal{E} that is an equivalence class of \sim for which $W \cap \bigcap \mathcal{E} = T$. For all $T \in \mathcal{T}$, set $Z_T = \prod_{S \in \varepsilon(T)} Y_S$. Let Z be the amalgam of $\langle Z_T \rangle_{T \in \mathcal{T}}$.

In the above definition, Z is homeomorphic to $\bigcup_{p \in W} \prod_{p \in S \in \mathcal{S}} Y_S$ with the subspace topology inherited from Y .

3. CONNECTIFIABLE AMALGAMS

Theorems 2.2 and 2.3 demonstrate similarities between products and amalgams. Of course, amalgams would not be very interesting if there were no major differences between them and products. Such differences arise for connectedness: unlike a product, an amalgam can be connected even if all its factors are not; connectedness of the base space is sufficient in most cases. Path-connectedness of an amalgam with a path-connected base space is harder to guarantee, but not by much. Some new positive connectification results fall out as corollaries.

Theorem 3.1. *Suppose X is connected (path-connected) and there is a finite $E \subseteq X$ such that for all $S \in \mathcal{S}$ we have $E \not\subseteq S$ or Y_S is connected (path-connected). Then Y is connected (path-connected).*

Proof. Let $y_0, y_1 \in Y$. It suffices to show that y_0 and y_1 are in the same component (path component). Let $E = \{p_0, \dots, p_{n-1}\}$ and set $p_n = \pi(y_1)$. Recursively define $z_0, \dots, z_{n+1} \in Y$ and $f_0, \dots, f_n \in \prod_{S \in \mathcal{S}} Y_S$ as follows. Set $z_0 = y_0$. Given z_i for some $i < n+1$, choose f_i such that it extends $z_i \cup (y_1 \upharpoonright (\text{dom } y_1 \setminus \text{dom } z_i))$. Set $z_{i+1} = f_i \upharpoonright \{S \in \mathcal{S} : p_i \in S\}$. For each $i < n+1$, let C_i be the amalgam of $\langle \{f_i(S)\}_{S \in \mathcal{S}} \rangle$, which is homeomorphic to X and therefore connected (path-connected). Then $z_i, z_{i+1} \in C_i$ for all $i < n+1$; hence, z_0, \dots, z_{n+1} are in the same component (path component).

Therefore, it suffices to show that z_{n+1} and y_1 are in the same component (path component). Since $\pi(z_{n+1}) = p_n = \pi(y_1)$, we have $\text{dom } z_{n+1} = \text{dom } y_1$. Set $\mathcal{A} = \text{dom } y_1$ and $\mathcal{B} = \{S \in \mathcal{A} : y_1(S) \neq z_{n+1}(S)\}$. Then $z_{n+1}, y_1 \in \prod_{S \in \mathcal{B}} Y_S \times \prod_{S \in \mathcal{A} \setminus \mathcal{B}} \{y_1(S)\}$; hence, it suffices to show that Y_S is connected (path-connected) for all $S \in \mathcal{B}$. Suppose $S \in \mathcal{B}$ and $E \not\subseteq S$. Then choose the least $i < n$ such that $p_i \notin S$. Then $S \not\subseteq \text{dom } z_{i+1}$. Choose the least $j < n+2$ such that $i+1 < j$ and $S \in \text{dom } z_j$. Then $z_j(S) = f_{j-1}(S) = y_1(S)$ because $S \not\subseteq \text{dom } z_{j-1}$. Hence, $z_j(S) = y_1(S) \neq z_{n+1}(S)$; hence $j < n+1$. Choose the least $k < n+2$ such that $j < k$ and $S \in \text{dom } z_k$ and $z_k(S) \neq y_1(S)$. If $S \not\subseteq \text{dom } z_{k-1}$, then $z_k(S) = y_1(S)$, which is absurd. Hence, $S \in \text{dom } z_{k-1}$; hence, $z_k(S) = z_{k-1}(S) = y_1(S)$, which is also absurd. Therefore, for all $S \in \mathcal{B}$, we have $E \subseteq S$; whence, Y_S is connected (path-connected). \square

Example 3.2. Suppose $X = [0, 1]$ and $\mathcal{S} = \{U \subseteq [0, 1] : U \text{ open}\}$ and $|Y_S| = 1$ for all $S \in \mathcal{S} \setminus \{[0, 1]\}$. Then Y is homeomorphic to the cone over $Y_{[0, 1]}$. If $1 \in S \in \mathcal{S}$, then $|Y_S| = 1$; hence, Theorem 3.1 implies Y is path-connected. Thus, Theorem 3.1 may be interpreted as constructing a class of generalized cones.

Corollary 3.3. Suppose $i \in \{1, 2, 3, 3\frac{1}{2}\}$ and X has a proper T_i connectification \tilde{X} and Y_S is T_i for all $S \in \mathcal{S}$. Then Y has a proper T_i connectification \tilde{Y} . If Moreover, if \tilde{X} is path-connected, then we may choose \tilde{Y} to be path-connected.

Proof. Fix $p \in \tilde{X} \setminus X$. For each $S \in \mathcal{S}$, let $\Phi(S)$ be an open subset of $\tilde{X} \setminus \{p\}$ such that $\Phi(S) \cap X = S$. Extend $\Phi : \mathcal{S} \rightarrow \mathcal{S}(\tilde{X})$ to some $\tilde{\mathcal{S}} \in \mathcal{S}(\tilde{X})$. For all $S \in \mathcal{S}$, set $\tilde{Y}_{\Phi(S)} = Y_S$. For all $S \in \tilde{\mathcal{S}} \setminus \Phi[\mathcal{S}]$, set $\tilde{Y}_S = 1$. Let \tilde{Y} be the amalgam of $\langle \tilde{Y}_S \rangle_{S \in \tilde{\mathcal{S}}}$. By Theorem 2.2, \tilde{Y} is T_i ; by Theorem 3.1, \tilde{Y} is connected, for $|\tilde{Y}_S| = 1$ if $p \in S \in \tilde{\mathcal{S}}$. Define $f : Y \rightarrow \tilde{Y}$ as follows. Given $y \in Y$, let $\pi(f(y)) = \pi(y)$; set $f(y)(\Phi(S)) = y(S)$ for all $S \in \text{dom } y$; set $f(y)(S) = 0$ for all $S \in \tilde{\mathcal{S}} \setminus \Phi[\text{dom } y]$ such that $\pi(y) \in S$. Then f is an embedding of Y into \tilde{Y} with dense range $\pi^{-1}X$; hence, \tilde{Y} is a proper T_i connectification of Y . Finally, by Theorem 3.1, \tilde{Y} is path-connected if \tilde{X} is. \square

The previously known result most similar to Corollary 3.3 is due to Druzhinina and Wilson: [4]: if all the path components of a T_2 (T_3 , metric) space are open and have proper pathwise connectifications, then the space has a T_2 (T_3 , metric) proper pathwise connectification.

Proof of Theorem 1.1. Every infinite product is an infinite product of countably infinite subproducts; every infinite topological sum is a countably infinite topological sum of topological sums. Moreover, products preserve the property of having a T_i pathwise connectification; topological sums preserve the T_i axiom and metrizability. Therefore, we only need to prove the theorem for all countably infinite products

of countably infinite topological sums. Set $X = \omega^\omega$ with the product topology. For each $m, n < \omega$, let $Z_{m,n}$ be a nonempty T_i space and let $S_{m,n} = \{p \in X : p(m) = n\}$; set $Y_{S_{m,n}} = Z_{m,n}$. Set $\mathcal{S} = \{S_{m,n} : m, n < \omega\} \in \mathbb{S}(X)$. Then clearly $Y \cong \prod_{m < \omega} \bigoplus_{n < \omega} Z_{m,n}$. Since $X \cong \mathbb{R} \setminus \mathbb{Q}$, there is a proper metrizable pathwise connectification of X , namely a copy of \mathbb{R} . By Corollary 3.3, Y has a proper T_i pathwise connectification. For the metrizable case, construct a connectification \tilde{Y} of Y as in the proof of Corollary 3.3, with \tilde{X} chosen to be homeomorphic to \mathbb{R} . Since \mathcal{S} is countable, the space \tilde{Y} is metrizable by Theorem 2.3. \square

If we care about connectedness but not path-connectedness, Theorem 3.1 and Corollary 3.3 can be considerably strengthened.

Theorem 3.4. *Suppose X is connected and either $X \notin \mathcal{S}$ or Y_X is connected. Then Y is connected.*

Proof. Let $y_0, y_1 \in Y$. It suffices to show y_1 is in the closure of the component of y_0 . Let U be a basic open neighborhood of y_1 . Then there exist $n < \omega$ and $\langle S_i \rangle_{i < n} \in (\text{dom } y_1)^n$ and $\langle U_i \rangle_{i < n}$ such that U_i is an open neighborhood of $y_1(S_i)$ for all $i < n$ and $U = \bigcap_{i < n} \pi_{S_i}^{-1} U_i$. Choose $f \in \prod_{S \in \mathcal{S}} Y_S$ such that f extends y_0 . Then there exists $E \subseteq X$ such that E is finite and $E \not\subseteq S$ for all $S \in \{S_i : i < n\} \setminus \{X\}$. For each $S \in \mathcal{S}$, set $Z_S = Y_S$ if Y_S is connected or $S \in \{S_i : i < n\}$; otherwise, set $Z_S = \{f(S)\}$. Let Z be the amalgam of $\langle Z_S \rangle_{S \in \mathcal{S}}$. Then Z is connected by Theorem 3.1. Moreover, $y_0 \in Z$ and $Z \cap U \neq \emptyset$. Thus, y_1 is in the closure of the component of y_0 . \square

Corollary 3.5. *Suppose $i \in \{1, 2, 3, 3\frac{1}{2}\}$ and X has a T_i connectification and Y_S is T_i for all $S \in \mathcal{S}$. Further suppose X has a proper T_i connectification or $X \notin \mathcal{S}$ or Y_X is connected. Then Y has a T_i connectification.*

Proof. If X has a proper T_i connectification, then so does Y by Corollary 3.3. If X is T_i and connected but has no proper T_i connectification, then Y is connected by Theorem 3.4. \square

4. A LARGE PATH-CONNECTED HOMOGENEOUS COMPACTUM

Definition 4.1. We say that a homogeneous compactum is *exceptional* if it is not homeomorphic to a product of dyadic compacta and first countable compacta.

In the previous section, we constructed a machine for strengthening connectification results. Next, we construct a machine that takes a homogeneous compactum and produces a path-connected homogeneous compactum. Applying this machine to a particular homogeneous compactum with cellularity \mathfrak{c} , we get a path-connected homogeneous compactum with cellularity \mathfrak{c} . Moreover, more careful analysis of the latter space's connectedness properties shows that it is exceptional.

All compact groups are dyadic, and most other known examples of homogeneous compacta are products of first countable compacta (see [13, 15]). Besides the exceptional homogeneous compactum we shall construct, there is, to the best of the author's knowledge, only one known example of an exceptional homogeneous compactum, and its existence is independent of ZFC. In [15], van Mill constructed a compactum K satisfying $\pi(K) = \omega$ and $\chi(K) = \omega_1$. Clearly, $\chi(Z) = \omega \leq \pi(Z)$ for all first countable spaces Z . Moreover, Efimov [5] and Gerlits [10] independently proved that $\pi\chi(Z) = w(Z)$ for all dyadic compacta Z . Hence, $\chi(Z) \leq \pi(Z)$ for

all Z homeomorphic to products of dyadic compacta and first countable compacta; hence, K is not homeomorphic to such a product. Under the assumption $\mathfrak{p} > \omega_1$ (which follows from $\text{MA} + \neg\text{CH}$), van Mill proved that K is homogeneous. However, van Mill also noted that all homogeneous compacta Z satisfy $2^{\chi(Z)} \leq 2^{\pi(Z)}$ as a corollary of a result of van Douwen [3]. In particular, if $2^\omega < 2^{\omega_1}$, then K is not homogeneous.

Definition 4.2. Given a topological space Z , let $\text{Aut}(Z)$ denote the group of autohomeomorphisms of Z . Let $\text{Aut}(Z)$ act on Z in the natural way: $gz = g(z)$ for all $z \in Z$ and $g \in \text{Aut}(Z)$. Let $\text{Aut}(Z)$ act on $\mathcal{P}(Z)$ such that $gE = g''E$ for all $E \subseteq Z$ and $g \in \text{Aut}(Z)$.

Lemma 4.3. *Let G be the stabilizer of \mathcal{S} in $\text{Aut}(X)$. Suppose Z is a homogeneous space and $Y_S = Z$ for all $S \in \mathcal{S}$. Further suppose G acts transitively on X . Then Y is homogeneous.*

Proof. Let $y_0, y_1 \in Y$. Choose $g \in G$ such that $g(\pi(y_0)) = \pi(y_1)$. Define $f: Y \rightarrow Y$ as follows. Given $y \in Y$, let $\pi(f(y)) = g(\pi(y))$ and $f(y)(gS) = y(S)$ for all $S \in \text{dom } y$. Then $f \in \text{Aut}(Y)$ because $f''(\pi_S^{-1}U) = \pi_{gS}^{-1}U$ and $f^{-1}(\pi_S^{-1}U) = \pi_{g^{-1}S}^{-1}U$ for all $S \in \mathcal{S}$ and U open in Z . Since $y_1, f(y_0) \in Z^{\text{dom } y_1}$, there exists $\langle h_S \rangle_{S \in \mathcal{S}} \in \text{Aut}(Z)^{\mathcal{S}}$ such that $(\prod_{S \in \text{dom } y_1} h_S)(f(y_0)) = y_1$. Let h be the amalgam of $\langle h_S \rangle_{S \in \mathcal{S}}$. Then $h \in \text{Aut}(Y)$ and $h(f(y_0)) = y_1$. Thus, Y is homogeneous. \square

Lemma 4.4. *Suppose X and Y_S are T_3 and $\text{ind } Y_S = 0$ for all $S \in \mathcal{S}$. Then $\text{ind } Y = \text{ind } X$.*

Proof. Set $n = \text{ind } X$. By (7) of Theorem 2.2, we may assume $n > 0$. We may also assume the lemma holds if X is replaced by a T_3 space with small inductive dimension less than n . First, Y is T_3 by Theorem 2.2. Next, given any $f \in \prod_{S \in \mathcal{S}} Y_S$, the amalgam of $\langle \{f(S)\} \rangle_{S \in \mathcal{S}}$ is homeomorphic to X ; hence, $\text{ind } Y \geq n$. Let $y \in Y$ and let U be a neighborhood of y . Then $y \in V_0 \subseteq U$ where $V_0 = \bigcap_{i < m} \pi_{S_i}^{-1}U_i$ for some $m < \omega$ and $\langle S_i \rangle_{i < m} \in (\text{dom } y)^m$ and $\langle U_i \rangle_{i < m}$ such that U_i is a clopen neighborhood of $y(S_i)$ for all $i < m$. Let W be a neighborhood of $\pi(y)$ such that $W \subseteq \bigcap_{i < m} S_i$ and $\text{ind } \partial W < n$. Set $V_1 = V_0 \cap \pi^{-1}W$.

It suffices to show that $\text{ind } \partial V_1 < n$. Set $V_2 = \pi^{-1}\partial W$. Then $\partial V_1 = V_0 \cap V_2$; hence, it suffices to show that $\text{ind } V_2 < n$. Let Z be the reduced amalgam of $\langle Y_S \rangle_{S \in \mathcal{S}}$ over ∂W . Then $Z \cong V_2$ and $\text{ind } Z = \text{ind } \partial W$ because $\text{ind } \partial W < n$ and every factor of Z , being a product of factors of Y , is zero-dimensional. \square

Theorem 4.5. *There is a path-connected homogeneous compact Hausdorff space Y with cellularity \mathfrak{c} , weight \mathfrak{c} , and small inductive dimension 1. Moreover, Y is not homeomorphic to a product of compacta that all have character less than \mathfrak{c} or have \mathfrak{c} a caliber. In particular, Y is exceptional.*

Proof. Let X be the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let \mathcal{S} be the set of open semicircles contained in X . Let γ be an indecomposable ordinal (i.e., not a sum of two lesser ordinals) strictly between ω and ω_1 . For each $S \in \mathcal{S}$, let Y_S be 2^γ with the topology induced by its lexicographic ordering. It is easily seen that Y_S is zero-dimensional compact Hausdorff and $w(Y_S) = c(Y_S) = \mathfrak{c}$. Moreover, Y_S is homogeneous [14]. Since $|\mathcal{S}| = \mathfrak{c}$, we have $w(Y) = c(Y) = \mathfrak{c}$. Moreover, Y is compact Hausdorff by Theorem 2.2. Since no $S \in \mathcal{S}$ contains a pair of antipodes, Y is path-connected by Theorem 3.1. The stabilizer of \mathcal{S} in $\text{Aut}(X)$ contains all

the rotations of X and therefore acts transitively on X ; hence, Y is homogeneous by Lemma 4.3. Also, by Lemma 4.4, $\text{ind } Y = \text{ind } X = 1$.

Seeking a contradiction, suppose Y is homeomorphic to a product of compacta that all have character less than \mathfrak{c} or have \mathfrak{c} a caliber. Then there exist a compactum Z with \mathfrak{c} a caliber, a sequence of nonsingleton compacta $\langle W_i \rangle_{i \in I}$ all with character less than \mathfrak{c} , and a homeomorphism φ from $Z \times \prod_{i \in I} W_i$ to Y . Clearly, W_i is path-connected for all $i \in I$. Choose $p \in X$. Then $\varphi^{-1}\pi^{-1}\{p\}$ is a G_δ -set; hence, there exist a nonempty $Z_0 \subseteq Z$ and $J \in [I]^{\leq \omega}$ and $q \in \prod_{j \in J} W_j$ such that $Z_0 \times \{q\} \times \prod_{i \in I \setminus J} W_i \subseteq \varphi^{-1}\pi^{-1}\{p\}$. Since $\pi^{-1}\{p\} = \prod_{p \in S \in \mathcal{S}} Y_S$, which is zero-dimensional, $Z_0 \times \{q\} \times \prod_{i \in I \setminus J} W_i$ is also zero-dimensional; hence, $\prod_{i \in I \setminus J} W_i$ is also zero-dimensional. Hence, W_i is not connected for all $i \in I \setminus J$; hence, $I = J$; hence, I is countable. Set $W = \prod_{j \in J} W_j$. Then $\chi(W) < \mathfrak{c}$ because $\text{cf } \mathfrak{c} > \omega$.

Let $H \subseteq X$ be an open arc subtending $\pi/2$ radians. Set $\mathcal{T} = \{S \in \mathcal{S} : H \subseteq S\}$. Then $|\mathcal{T}| = \mathfrak{c}$. Choose a nonempty open box $U \times V \subseteq Z \times W$ such that $U \times V \subseteq \varphi^{-1}\pi^{-1}H$ and $U = \bigcup_{n < \omega} U_n$ where U_n is open and $\overline{U}_n \subseteq U_{n+1}$ for all $n < \omega$. Choose $r \in V$ and set $\kappa = \chi(r, W) < \mathfrak{c}$. Let $\langle V_\alpha \rangle_{\alpha < \kappa}$ enumerate a local base at r . By compactness, we may choose, for each $\alpha < \kappa$ and $n < \omega$, a finite set $\sigma_{n,\alpha}$ of basic open subsets of Y such that $\overline{U}_n \times \{r\} \subseteq \varphi^{-1} \bigcup \sigma_{n,\alpha} \subseteq U_{n+1} \times V_\alpha$. Set $G = \bigcup_{n < \omega} \bigcap_{\alpha < \kappa} \bigcup \sigma_{n,\alpha}$. Since $\kappa < \mathfrak{c}$, there exist nonempty $\mathcal{R} \subseteq \mathcal{T}$ and $E \subseteq \bigcup_{x \in H} \prod_{x \in S \in \mathcal{S} \setminus \mathcal{R}} Y_S$ such that $G = E \times \prod_{S \in \mathcal{R}} Y_S$. Hence, $c(G) = \mathfrak{c}$. Since $\varphi^{-1}G = U \times \{r\}$, we have $c(U) = \mathfrak{c}$. Since U is an open subset of Z , we have $c(Z) \geq \mathfrak{c}$, which yields our desired contradiction, for $\mathfrak{c} \in \text{cal}(Z)$. \square

Remark 4.6. If there is a homogenous compactum with cellularity $\kappa > \mathfrak{c}$ (that is, if van Douwen's Problem (see [13]) has a positive solution), then the proof of Theorem 4.5 is easily modified to produce a path-connected homogeneous compactum with cellularity κ .

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